Note: Lie Group and Lie Algebra

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Abstract

Here is the brief note of Lie Group and Lie Algebra taught in Hunan University by Prof. Zhen FANG. Enjoy the study !

Contents

1 Groups: Discrete or Continuous, Finite or Infinite

Direct product of groups $H \equiv F \otimes G$. If $f \in F$, $g \in G$, then $(f, g) \in H$.

- Multiplication: $(f, g)(f', g') = (ff', gg')$;
- Identity: $I_H = (I_F, I_G)$;
- Inverse: $(f, g)^{-1} = (f^{-1}, g^{-1})$;
- $|F| = m$, $|G| = n$, $|H| = mn$;

A map $f : G \to G'$ of a group into the group G' is called a homomorphism if it preserves the multiplicative structure of G, that is, $f(g_1)f(g_2) = f(g_1 g_2)$. A homomorphism becomes an isomorphism if the map is one-to-one and onto.

1.1 Finite groups

Example 1.1 Finite groups

1. Cyclic groups: $Z_k = \left\{I, g, g^2, \cdots, g^{k-1}\right\}$ with $g^k = I$;
2. Permutation group S_n: all permutations of n objects;
3. Alternating group A_n: even permutations of n objects;
4. Quaternion group: $\mathcal{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$

Theorem 1.2 Lagrange's theorem
If group H with m elements is a subgroup of G with n elements, then m is a factor of n.

Theorem 1.3 Cayley's theorem
Any finite group G with n elements is isomorphic (that is, identical) to a subgroup of the permutation group S_n.

Theorem 1.4 Square root of the identity
Let G be a finite group of even order (G has an even number of elements). There exists at least one element $g \neq I$ such that $g^2 = I$.

The set of transformations that leave the n-sided regular polygon invariant form the dihedral group D_n.

$$D_n = \left\langle R, r | R^n = I, r^2 = I, rRr = R^{-1} \right\rangle$$

Two generators: the roration R through $2\pi/n$ and the reflection r through a median.

$$rRr = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R^{-1}$$

1.1.1 Equivalence class

In a group G, two elements g and g' are equivalent ($g \sim g'$) if there exists another element f such that $g' = f^{-1}gf$. The transformation $f^{-1}gf$ is also called the conjugate of g. If g and g' are equivalent, we also say that g is conjugate to g'

Proposition 1.8 Facts about equivalent class

- Every element in an abelian group constitutes a class;
- The identity itself constitutes a class in any group;
- Given a class $c = \{g_1, \cdots, g_{n_c}\}$, the inverse $\bar{c} = \left\{g_1^{-1}, \cdots g_{n_c}^{-1}\right\}$ forms a class;

1.1.2 Invariant subgroup, simple group and quotient group

Let $H = \{h_1, h_2, \cdots\}$ be a subgroup of G. Then $g^{-1}Hg$ is also a subgroup of G. If $g^{-1}Hg = H$ for all $g \in G$, then H is called an invariant subgroup or normal subgroup of G, which is denoted by $H \lhd G$ or $G \rhd H$.
A group is called a simple group if it does not have any proper invariant subgroup.

Theorem 1.11 The kernel of a homomorphism of G is an invariant subgroup of G
Let f be a homomorphic map of a group G into itself: $f(g_1)f(g_2) = f(g_1g_2)$. The kernel of f, that is, the set of elements that are mapped to the identity, is an invariant subgroup.

Proposition 1.12 The cosets of an invariant subgroup form the quotient group
Let $H = \{h_1, h_2 \cdots\}$ be an invariant subgroup of G. Then the left coset of H is equal to its right coset ($Hg = gH$). The series of cosets $\{g, H | g_i \in G\}$ form a group called quotient group denoted by $Q = G/H$. For example $Z_2 = \mathcal{Q}/Z_4$. We have $N(Q) = N(G)/N(H)$. Note that Q is not a subgroup of G in general.
Proof Close under multiplication $(g_aH)(g_bH) = (g_ag_bH) since (\mathrm{g}_a h_i)(g_bh_j) = g_a(g_bg_b^{-1})h_ig_bh_j = g_ag_b(g_b^{-1}h_ig_b)h_j = (g_ag_b)(h_lh_j)$ with $h_l = g_b^{-1}h_ig_b$. The identity is $IH = H$ and the inverse of gH is $g^{-1}H$.

Derived subgroup Define $(a, b \in G)$

$$\langle a, b\rangle \equiv a^{-1}b^{-1}ab = (ba)^{-1}(ab)$$

Denote by $\{x_1, x_2, \cdots\}$ the objects $\langle a,b\rangle$ with $a, b \in G$. These objects (maybe some of them) constitute a subgroup of G, known as the derived subgroup $\mathcal{D}$. The identity is $\langle a,a\rangle = I$ and the inverse $\langle a,b\rangle^{-1} = \langle b,a\rangle$.

(Note that the product $\langle a,b\rangle \langle c,d\rangle$ need not have the form $\langle e,f\rangle$ for some $e, f \in G$, and derived subgroup $\mathcal{D}$ is not necessarily equal to the set of all objects of the form $\langle a,b\rangle$ as a and b range over G.)

The object $\langle a,b\rangle \equiv a^{-1}b^{-1}ab = (ba)^{-1}(ab)$ measures how much ab differs from ba. In other words, the derived subgroup tells us how non-abelian the group G is. The larger $\mathcal{D}$ is, the farther away G is from being abelian, roughly speaking. Conversely, $\mathcal{D} = I$ for abelian groups.

Example 1.13

1. The derived subgroup of S_n, A_n;
2. The derived subgroup of A_4, $V = Z_2 \otimes Z_2$;
3. The derived subgroup of $\mathcal{Q}$, $Z_2 = \{1, -1\}$;
4. The derived subgroup of Z_4, I;

Proposition 1.14

$\mathcal{D}$ is an invariant subgroup of G.

Theorem 1.15

Given a group G and one of its invariant subgroups H, form the quotient group $Q = G/H$. Suppose that Q has no invariant subgroup, then H is the maximal invariant subgroup.

2 Group Representation

Given a group, the idea of a representation is to associate each element g with a $d \otimes d$ matrix $D(g)$ such that

$$D(g_1)D(g_2) = D(g_1g_2)$$

which "reflects" the multiplicative table of the group. $D(I) = I_d$ and $D(g^{-1}) = D(g)^{-1}$.

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Q: Does every group G have a representation?

A: Every finite group can be represented by matrices since every finite group is isomorphic to a subgroup of S_n.

number of **reducible** representations: infinite;
number of **irreducible** representations: finite.

Trivial, faithful, unfaithful, defining or fundamental representations
○ In representation theory of Lie groups and Lie algebras, a fundamental representation is an irreducible finite-dimensional representation of a semisimple Lie group or Lie algebra whose highest weight is a fundamental weight. For example, the defining module of a classical Lie group is a fundamental representation. Any finite-dimensional irreducible representation of a semisimple Lie group or Lie algebra can be constructed from the fundamental representations by a procedure due to Elie Cartan. Thus in a certain sense, the fundamental representations are the elementary building blocks for arbitrary finite-dimensional representations.

Character of a representation: We write $D^{(r)}(g)$ for the matrix representing the element g in the representation r. The character $\chi^{(r)}$ of the representation is defined by

$$\chi^{(r)}(g) \equiv \mathrm{tr} D^{(r)}(g)$$

Character is a function of equivalent class: if $g_1 \sim g_2$, then $\chi^{(r)}(g_1) = \chi^{(r)}(g_2)$.

Theorem Unitary representations and unitary theorem for finite groups
Finite groups have unitary representations, $D^\dagger(g)D(g) = I$ for all g and all representations.
Suppose H is

$$H = \sum_g \tilde{D}^\dagger \tilde{D}$$

then diagonalize it:

$$\rho^2 = W^\dagger H W = \sum_g \left(\tilde{D}(g)W\right)^\dagger \tilde{D}(g)W$$

is diagonal, real and positive. Then $D(g) \equiv \rho W^\dagger \tilde{D}(g) W \rho^{-1}$ is unitary.

Theorem Unitary theorem for compact groups:
The representations of compact groups are unitary.

Product representation: The direct product of two representations r and s of a group G with representation matrices $D^{(r)}(g)$ and $D^{(s)}(g)$ of dimensions d_r and d_s respectively is also a representation of G of dimension $d_r d_s$, which is called the **(direct) product representation** and is denoted by $r \otimes s$.

$$D(g) = D^{(r)}(g) \otimes D^{(s)}(g)$$

$$D(g)_{a\alpha, b\beta} = D^{(r)}(g)_{ab} \otimes D^{(s)}(g)_{\alpha\beta}$$

Proof

$$D(g)D(g') = \left(D^{(r)}(g) \otimes D^{(s)}(g)\right)\left(D^{(r)}(g') \otimes D^{(s)}(g')\right) = \left(D^{(r)}(g)D^{(r)}(g')\right) \otimes \left(D^{(s)}(g)D^{(s)}(g')\right) = D(gg')$$

An important question is how the product representation reduces to a direct sum of irreducible representations. (See next chapiter)

The character of the direct product representation of two representations $D^{(r)}(g)$ and $D^{(s)}(g)$ is

$$\chi(c) = \sum_{a\alpha} D(g)_{a\alpha,a\alpha} = \left(\sum_a D^{(r)}(g)_{aa}\right)\left(\sum_\alpha D^{(s)}(g)_{\alpha\alpha}\right) = \chi^{(r)}(c)\chi^{(s)}(c)$$

3 Schur's Lemma

Schur's Lemma
Given an irreducible representation $D(g)$ of a finite group G, if there is some matrix A such that $AD(g) = D(g)A$ for all g, then $A = \lambda I$ for some constant λ.

A lemma to Schur's Lemma
Given an irreducible representation $D(g)$ of a finite group G, if there is some Hermitian matrix H such that $HD(g) = D(g)H$ for all g, then $H = \lambda I$ for some constant λ.

Since $D(g)$ is unitary, the above two statements are equivalent.

- Any matrix can be written as a sum of two hermitian matrices

$$A = \frac{1}{2}\left[(A + A^\dagger) - ii(A - A^\dagger)\right]$$

- then

$$AD(g) = D(g)A \leftrightarrow D(g)^\dagger A^\dagger = A^\dagger D(g)^\dagger \leftrightarrow A^\dagger D(g) = D(g)A^\dagger$$

  so if there is some matrix A such that $AD(g) = D(g)A$ for all g, then $(A + A^\dagger)D(g) = D(g)(A+A^\dagger)$ and $i(A-A^\dagger)D(g) = D(g)i(A-A^\dagger)$. According to the lemma, $A+A^\dagger = \lambda_1 I$ and $i(A - A^\dagger) = \lambda_2 I$. Thus $A = \frac{1}{2}(\lambda_1 - i\lambda_2)I$.

A little lemma to a lemma to Schur's Lemma
Given an irreducible representation $D(g)$ of a finite group G, if there is some diagonal matrix

M such that $MD(g) = D(g)M$ for all g, then $M = \lambda I$ for some constant λ.
Proof (M is diagonal)

$$[MD(g)]^i_j = M^i_i D^i_j(g) = [D(g)M]^i_j = D^i_j(g) M^j_j$$

so for all $g \in G$, all elements of M's diagonal equal.

$$\left(M^i_i - M^j_j\right) D^i_j(g) = 0$$

- Key words for Schur's Lemma: irreducible representation, finite group, some matrix, all g;
- Schur's Lemma means that you can't find a matrix apart from the identity matrix that commutes with all the representation matrices $D(g)$ of an irreducible representation of a finite gorup;
- Since all the representations of a finite (compact) group can be made unitary according to the unitary theorem, we just assume that $D(g)$ is unitary;

4 The Great Orthogonality Theorem

The great orthogonality theorem is the central theorem of representation theory.

Lemma
Given a d-dimensional irreducible representation $D(g)$ of a finite group G, and define $A = \sum_g D^\dagger(g) X D(g)$ for some arbitrary matrix X, we have $\lambda = \frac{N(G)}{d}\mathrm{tr}X$

$$A = \sum_g D^\dagger(g) X D(g) = \lambda I_d$$

Proof

$$D^\dagger(g) A D(g) = D^\dagger(g)\left(\sum_{g'} D^\dagger(g') X D(g')\right) D(g) = \left(\sum_{g'} D^\dagger(g'g) X D(g'g)\right) A$$

with Schur's lemma, $A = \lambda I_d$, so

$$\mathrm{tr}A = \lambda d = \sum_g \mathrm{tr}D^\dagger(g) X D(g) = \sum_g \mathrm{tr}X = N(G)\mathrm{tr}X \qquad \Rightarrow \qquad \lambda = \frac{N(G)}{d}\mathrm{tr}X$$

Theorem The Great Orthogonality Theorem: (select X as an identity matrix)
Given a d-dimensional irreducible representation $D(g)$ of a finite group G, we have

$$\sum_g D^\dagger(g)^i_j D(g)^k_l = \frac{N(G)}{d}\delta^i_l \delta^k_j$$

with $N(G)$ the number of group elements.

Obviously, we select $X_k^j = 1$ in the lemma. And it's necessary to verify that D(g) is irreducible in order to use Schur's lemma.

Proposition

If r and s are two inequivalent representations, then

$$\sum_g D^{(r)\dagger}(g)^i_j D^{(s)}(g)^k_l = 0$$

Proof define a $d_r \times d_s$ matrix

$$A = \sum_g D^{(r)\dagger}(g) X D^{(s)}(g)$$

with X an arbitrary $d_r \times d_s$ matrix. Using the unitarity of the two representations, we have $D^{(r)\dagger}(g) A D^{(s)}(g) = A$, thus for any $g \in G$

$$A^\dagger D^{(r)}(g^{-1}) = A^\dagger D^{(r)\dagger}(g) = D^{(s)}(g)^\dagger A^\dagger = D^{(s)}(g^{-1}) A^\dagger$$

$$AA^\dagger D^{(r)}(g^{-1}) = AD^{(s)}(g^{-1}) A^\dagger = D^{(r)}(g^{-1}) AA^\dagger$$

if $d_r \neq d_s$, $AA^\dagger = \lambda I_{dr} \quad \Rightarrow \quad \lambda = 0$;
if $d_r = d_s$, so $\det A = 0$ and then $\lambda = 0$; ($\det A \neq 0$ contradicts the presumption)

A more general form of the Great Orthogonality theorem

Given two irreducible representations r and s of a finite group G with d_r and d_s dimensions, respectively, and let $D^{(r)}(g)$ and $D^{(s)}(g)$ be the representation matrices, we have

$$\sum_g D^{(r)\dagger}(g)^i_j D^{(s)}(g)^k_l = \frac{N(G)}{d_r} \delta^{rs} \delta^i_l \delta^k_j$$

with $N(G)$ the number of group elements.

For each triplet (s, k, l), regard the array of complex number $D^{(s)}(g)^k$ as a vector in an $N(G)$-dim complex vector space. So $\sum_s d_s^2$ vectors re orthogonal to each other. Since there are at most $N(G)$ independent vectors in an $N(G)$-dimensional complex vector space, thus we have $\sum_s d_s^2 \leqslant N(G)$. Intuitively, the irreducible reprensentations of a group of a certain "size" $N(G)$ can't be "too big".

Eq. above leads to a reduced constraining on the nontrivial representation matrices:

$$\sum_g D^{(r)\dagger}(g)^i_j = 0$$

which is obtained by taking s to be the trivial representation and let $k = l$ and sum it.

Corollary Orthogonality theorem for characters

Let $\chi^{(r)}(c)$ and $\chi^{(s)}(c)$ be characters of two irreducible representations $D^{(r)}(g)$ and $D^{(s)}(g)$ of a *finite group* G with d_r and d_s dimensions respectively, we have

$$\sum_c n_c \left[\chi^{(r)}(c)\right]^* \chi^{(s)}(c) = N(G)\delta^{rs}$$

with n_c denoting the number of elements belonging to class c.

We could construct a **character table** displaying $\chi^{(r)}(c)$ of a finite group G with $N(C)$ row representing different irreducible representations r and $N(R)$ columns representing different equivalence classes c. Take A_4 as an example, ($\omega = \exp(2\pi i/3)$)

A_4	n_c	c	1	1'	1"	3
	1	I	1	1	1	3
Z_2	3	(12)(34)	1	1	1	-1
Z_3	4	(123)	1	ω	ω^*	0
Z_3	4	(132)	1	ω^*	ω	0

which satisfies the orthogonal theorem.

5 Real, Pseudo-real and Complex Representations

Conjugate representation

Let $D(g)$ furnish an irreducible representation r of a group G. Then $D(g)^*$ also forms a representation of G. $D(g_1)^*D(g_2)^* = D(g_1g_2)^*$, which is known as the conjugate of r denoted by r^*, or the conjugate representation of G with respect to r. The characters of the representation r^* is the complex conjudate of the characters of r:

$$\chi^{(r^*)}(c) = \mathrm{tr}D(g)^* = [\mathrm{tr}D(g)]^* = \chi^{(r)}(c)^*$$

Complex & noncomplex representations

The question whether the two representations r and r^* are equivalent or not leads naturally to be complex and noncomplex representations. If r and r^* are equivalent, if there is a matrix S such that (for all $g \in G$)

$$D(g)^* = SD(g)S^{-1}$$

we say r and r^* is **noncomplex**; otherwise **complex**.

Using characters to determine whether a representation is complex or noncomplex

- if a character $\chi^{(r)}(c)$ is complex, $\chi^{(r)}(c)^* \neq \chi^{(r)}(c)$, then the representation r is complex;
- if the representation r is noncomplex, then all the characters $\chi^{(r)}(c)$ are real: $\chi^{(r)}(c)^* = \chi^{(r)}(c)$ for all c.

Proof Note that for noncomplex representation the characters of the conjudate representation R^* are equal to the characters of r:

$$\chi^{(r^*)}(c) = \text{tr}D(g)^* = \text{tr}SD(g)S^{-1} = \chi^{(r)}(c)$$

Constraints on S for noncomplex representations
Suppose that $D(g)$ furnishes a noncomplex irreducible representation r of a group G, then the matrix S of similarity transformation satisfies the following properties:

- S is either symmetric of anti. if S is symm , we say that the irreducible representation r is real;
- if S is antisymmetric ($S^T = -S$), we say that the irreducible representation r is **pseudoreal**;
- S is unitary $S^\dagger S = I$.

Proof r is noncomplex and unitary

$$D(g)^* = SD(g)S^{-1}$$

Lemma The square root of a unitary symmetric matrix is also unitary symmetric.
Given a unitary symmetric matrix U, there exists a unitary symmetric matrix W such that $W^2 = U$.

Proposition Real representation is really real
If an irreducible representation is real, then there is some basis in which all the entries of the matrices $D(g)$ are real.
Proof suppose that there is a unitary symmetric matrix W such that $W^2 = S$, we have $W^{-1} = W^\dagger = W^*$ and

$$W^2D(g)W^{-2} = D(g)^* \quad \Rightarrow \quad WD(g)W^{-1} = W^{-1}D(g)^*W = W^*D(g)^*(W^{-1})^* = (WD(g)W^{-1})^*$$

Theorem
An *invariant bilinear* exists if and only if the irreducible representation is *noncomplex* (real or pseudo-real). More precisely, ($D(g)$ unitary)

- Let r be a noncomplex irreducible representation r and S be the similarity transformation matrix relating the two conjugate representations r and r^* by $D^*(g) = SD(g)S^{-1}$, then $y^T Sx$ is an invariant bilinear.
- Conversely, if $y^T Sx$ is invariant, then the two irreducible conjugate representation furnished by $D(g)$ and $D^*(g)$ must be noncomplex (or equivalent), i.e. $D^*(g) = SD(g)S^{-1}$.

Proof

$$D^*(g) = SD(g)S^{-1} \leftrightarrow D^*(g)S = SD(g) \leftrightarrow S = D^T(g)SD(g) \leftrightarrow y^T Sx \to y^T D(g)^T SD(g)x = y^T Sx$$

The inverse of the above proof is clearly true. In the above proof we did not use the unitary (anti-)symmetric property of S, just used the unitarity of $D(g)$. The core is to prove $S = D^T(g)SD(g)$.

Given an irreducible representation r furnished by $D(g)$, we can construct a matrix S of the form

$$S \equiv \sum_{g\in G} D(g)^T X D(g)$$

for an arbitrary X such that $y^T Sx$ is an invariant bilinear automatically (since $S = D^T(g)SD(g)$).

Proposition Sum of the characters of squares in **complex** irreducible representation

1. If the irreducible representation is **complex**, then we have, for arbitary X,

$$S \equiv \sum_{g\in G} D(g)^T X D(g) = 0$$

2. If the irreducible representation is complex, then (with $X_{il} = 1$ only)

$$\sum_{g\in G} D(g)_{ij} D(g)_{lk} = 0$$

3. If the irreducible representation is complex, then

$$\sum_{g\in G} D(g^2)_{ik} = 0$$

4. If the irreducible representation is complex, then

$$\sum_{g\in G} \chi(g^2) = 0$$

Proposition Sum of the characters of squares in noncomplex irreducible representation

- The irreducible representation r is noncomplex (real or pseudo-real) *iff* for some X.

$$S \equiv \sum_{g\in G} D(g)^T X D(g) \neq 0$$

- If the irreducible representation r is noncomplex, then

$$\sum_{g\in G} \chi(g^2) = \eta \sum_{g\in G} \chi(g)\chi(g) = \eta N(G)$$

with $\eta = \pm 1$. For real representations, $\eta = 1$; for pseudo-real representations, $\eta = -1$.

Thus, we have a criteria on the reality of the irreducible representation by the sum of characters.

Written in another form

$$\sum_{g\in G} \sigma_f \chi^{(r)}(f) = \eta^{(r)} N(G)$$

$\eta = 1$, real; $\eta = -1$, pseudo-real; $\eta = 0$, complex; where σ_f is the number of square roots (that belong to G) of each $f \in G$, that is, the number of different solutions $g \in G$ to the equation $g^2 = f \in G$. Note that some σ_f might be 0.

Proposition The number of square roots in a group G

Given the irreducible representation r (or the character table) of a group G, the number of square roots of each element $f \in G$ is determined by the formula

$$\sigma_f = \sum_r \eta^{(r)} \chi^{(r)}(f) = \sum_{r\in\mathbb{R}} \chi^{(r)}(f) - \sum_{r\in pseudo} \chi^{(r)}(f)$$

Particularly, the number of square roots of the identity is

$$\sigma_I = \sum_r \eta^{(r)} d_r = \sum_{r\in\mathbb{R}} d_r - \sum_{r\in pseudo} d_r$$

Proof $(f \in c, f' \in c')$

$$\sum_r \eta^{(r)} \chi^{(r)*}(f') N(G) = \sum_r \left(\sum_{f\in G} \sigma_f \chi^{(r)}(f)\right) \chi^{(r)*}(f') = \sum_{f\in G} \sigma_f \left(\sum_r \chi^{(r)}(f) \chi^{(r)*}(f')\right)$$
$$= \sum_{f\in G} \sigma_f \frac{N(G)}{n_c} \delta_{cc'} = \sigma_{c'} N(G)$$

The sum of the representation matrices of squares in complex or noncomplex irreducible representations

Given an irreducible representation r furnished by $D(g)$, the sum of the representation matrices of squares in this irreducible representation is

$$\sum_g D^{(r)}(g^2) = \left(\frac{\eta^{(r)}}{d_r} N(G)\right) I$$

with $\eta^r = 1$, real; $\eta^r = -1$, pseudo-real; $\eta^r = 0$, complex.

Proof We construct another matrix A of the form $A \equiv \sum_g D(g^2)$

$$D^{-1}(g')AD(g') = D^{-1}(g')\left(\sum_g D(g^2)\right)D(g') = \sum_g D(g'^{-1}gg'g'^{-1}gg') = A$$

by Schur's lemma, we have $A = cI$ with c determined by taking the trace:

$$\sum_g \mathrm{tr}D^{(r)}(g^2) = cd_r = \sum_g \chi^{(r)}(g^2) = \sum_f \sigma_f \chi^{(r)}(f) = \sum_s \eta^{(s)} \sum_f \chi^{(s)*}(f)\chi^{(r)}(f) = \eta^{(r)}N(G)$$

Sum of characters of the product of squares in complex or noncomplex irreducible representations

$$\sum_f \sum_g \chi^{(r)}(f^2g^2) = N(G)\left(\eta^{(r)}/d_r\right)\sum_f \chi^{(r)}(f^2) = \left(N(G)\eta^{(r)}\right)^2/d_r$$

$$\sum_h \tau_h \chi^{(r)}(h) = \left(N(G)\eta^{(r)}\right)^2/d_r$$

where τ_h denotes the number of solutions of $f^2g^2 = h^2$ for each $h \in G$. Multiplying by $\chi^{(r)*}(h')$ and summing over r:

$$\sum_r \left(\sum_h \tau_h \chi^{(r)}(h)\right)\chi^{(r)*}(h') = N(G)^2 \sum_r \left[\left(\eta^{(r)}\right)^2/d_r\right]\chi^{(r)*}(h')$$
$$= \sum_h \tau_h \left(\sum_r \chi^{(r)}(h)\chi^{(r)*}(h')\right) = \sum_h \tau_h \frac{N(G)}{n_c}\delta_{cc'} = \tau_{h'}N(G)$$

we get

$$\tau_h = N(G)\sum_r \left(\eta^{(r)}\right)^2 \chi^{(r)}(h)/d_r$$

Particularly

$$\tau_I = N(G)\sum_r \left(\eta^{(r)}\right)^2 = N(G)\left(\sum_{r\in\mathbb{R}} 1 + \sum_{r\in pseudo} 1\right)$$

remark that this simple formula determines the number of solutions to the equation $f^2g^2 = I$ in group G.

6 Symmetry, Representation, Degeneracy in Quan. Mech.

Proposition

1. In quan mech, transformations are realized as unitary operators T.

2. A set of transformations T s(that we assume to be closed with multiplication) how the Hamiltonian H invariant form a symmetry group G.

$$(T_1T_2)^\dagger H(T_1T_2) = H$$

3. If T leaves H invariant (i.e. $T^\dagger HT = H$), its action on ψ produces an eigenstate of H with the same energy E.

$$H(T\psi) = HT\psi = TH\psi = TE\psi = E(T\psi)$$

Remark

1. Under a transformation T, the Hamiltonian H as an operator transforms by $H^\dagger = THT^\dagger$
2. Since T is unitary, a transformation T leaves the Hamiltonian H invariant if and only if T commutes with H
3. One should be aware that when we say a quantum system has a symmetry (or is invariant under a symmetry group G), it is equivalent to say that the Hamiltonian H is invariant under the transformations Ts of G. Since

$$H' = THT^\dagger \Rightarrow H'\psi' = E\psi' \Rightarrow \psi' = T\psi$$

the invariance of H under T implies that the eigenvalue equations have the same form before and after the transformation T (i.e. the system is invariant ...)

Proposition

If a quantum system has a d-field degeneracy, these should be a symmetry hiding in this system, and the d degenerate eigenstates ψ furnish a d-dimensional irreducible representation of the symmetry group G.

$$\psi_a \to \psi'_a = D(T)_{ab}\psi_b \qquad D(T_2T_1) = D(T_2)D(T_1)$$

Conversely, if a quantum system is invariant under a symmetry group G, the irreducible representations of G determine the possible degeneracies of the system.

Proposition

Let $\psi_a, a = 1, \cdots, d$ be solutions of $H\psi_a = E_a\psi_a$. If the states ψ_a form a d-dimensional irreducible representation of a symmetry group G with $T\psi_a = D(T)_{ab}\psi_b$, then ψ_a have the same energy $E_a = E$.

Proof The fact that ψ_a form an (irreducible) representation of G implies that $THT^\dagger = H$

$$TH\psi_a = THT^\dagger T\psi_a = HT\psi_a = D(T)_{ab}H\psi_b = D(T)_{ab}E_b\psi_b$$

$$TH\psi_a = E_aT\psi_a = E_aD(T)_{ab}\psi_b \qquad \Rightarrow \qquad D(T)_{ab}E_b = E_aD(T)_{ab}$$

which could be written as the matrix form $D(T)\mathcal{E} = \mathcal{E}D(T)$ with $\mathcal{E} = EI_d$.

Consider a quantum system with Z_2 symmetry

$$\psi' = S\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_1 + \psi_2 \\ \psi_1 - \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi(x) + \psi(-x) \\ \psi(x) - \psi(-x) \end{pmatrix} \equiv \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

note that $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ combines two diagonal matrices. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$...

Proposition 4.5 The irreducible tensor representation of SO(3)

1. Besides the trivial representation, the totally symmetric traceless tensors

$$(with$$

S^{ij} being vectors that form the vector (or defining) representation) furnish all the irreducible tensor representations of SO(3)...

... 2. The number of all possible ways of 1,2,3 distributed on the indices of $S^{i_1 i_2 \cdots i_j}$ is

$$\sum_{k=0}^{j}(k+1) = \frac{1}{2}(j+1)(j+2)$$

Proposition 4.6 The (real) irreducible tensor representations of SO(2) 1. Besides the trivial representation, the totally symmetric traceless tensors Sxxx () furnish all the (real) irreducible tensor representations of SO(2), which are denoted by js. 2. The dimensions of the (real) irreducible tensor representations of SO(2) furnished by Sxxx

Polar deccomposition....

When not restricted to real representations, the representation given in 4.20 is in fact reducible,

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \cos j\theta & \sin j\theta \\ -\sin j\theta & \cos j\theta \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} \exp(ij\theta) & 0 \\ 0 & \exp(-ij\theta) \end{pmatrix}$$